

EMBEDDING ORDERS INTO THE CARDINALS WITH DC_κ

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ABSTRACT. Jech proved that every partially ordered set can be embedded into the cardinals of some model of ZF . We extend this result to show that every partially ordered set can be embedded into the cardinals of some model of $\text{ZF} + \text{DC}_{<\kappa}$ for any regular κ . We use this theorem to show that for all κ , the assumption of DC_κ does not entail that there are no decreasing chains of cardinals. We also show how to extend the result to and embed into the cardinals a proper class which is definable over the ground model. We use this extension to give a large cardinals-free proof of independence of the weak choice principle known as **WISC**.

1. INTRODUCTION

Assuming the axiom of choice, cardinals trivially form a well-ordered class, but with its failure their order structure can be as complex as desired. An interesting example for this range of possibilities is Jech's theorem in which he proves that if (P, \leq) is a partial order then there exists a model of **ZFA** (Zermelo-Fraenkel with Atoms) in which (P, \leq) can be embedded into the cardinals (see [Jec66]). The theorem was complemented by the Jech-Sochor embedding theorem which allowed to carry the consistency result into **ZF** and remove the need for atoms (see [JS66b, JS66a] and [Jec73, Chapter 6]). This theorem tells us, essentially, that there are no limitations on the order structure of cardinals defined by injections.

Jech's original proof included adding many counterexamples to **DC** (in the form of Dedekind-finite sets). While it can be modified to allow DC_κ to hold, the Jech-Sochor theorem is not suited to transfer universal statements such as DC_κ . We should point out that Pincus improved upon Jech-Sochor's original work and showed that it is possible to transfer injectively boundable statements (a class of statements which include DC_κ), and more. In this paper we give a direct forcing argument to Jech's proof, and this allows us to preserve DC_κ up to an arbitrary (but fixed) cardinal κ . We then proceed to show that in fact ordered classes (which are definable in the ground model) may be embedded into the cardinals while preserving DC_κ .

The authors of [BM90] remark that it is unknown whether or not the assumption that there are no decreasing sequences of cardinals (in the \leq relation) implies that the axiom of choice holds. They write in section four: "The answer [...] is almost certainly negative, but thus far there is no proof." and mention that there has been some disagreement on the topic in the past. We will use the improved embedding theorem to show that for any κ it is consistent with $\text{ZF} + \text{DC}_\kappa$ that there are decreasing chains of cardinals. While there is no positive results yet, this somehow suggests that the axiom of choice might be equivalent to the assertion "There is no infinite decreasing sequence of cardinals".

Decreasing sequences of cardinals also have importance in answering a question of Feldman & Orhon which appeared in [FOB08]. In the paper the authors prove ¹

Date: January 1, 2013.

2000 Mathematics Subject Classification. Primary 03E25; Secondary 03E35.

¹The original proof is due to Tarski, see [RR85, Form T3(n), pp. 22-23].

that for any $k \in \omega \setminus \{0, 1\}$ the assertion that “Every antichain of cardinals has less than k members” implies the axiom of choice, and a question asks about replacing the finite bound by ω . Feldman and Orhon conjectured that “Every antichain of cardinals is finite” does not imply the axiom of choice in ZF. The question is still open, but we will show that $\text{ZF} + \text{DC}_\kappa$ cannot prove that every antichain is finite.

In [Rog90] the author proves that it is consistent relative to the consistency of ZF that there exists an antichain of cardinals which is not a set. We extend this result and show its compatibility with DC_κ . We use this extension to show the consistency of long chains and antichains of cardinals, and to give a large cardinals-free consistency result of the failure of WISC, a recent choice principle related to constructive set theory.

2. BASIC DEFINITIONS

Suppose that \mathfrak{M} is a countable transitive model of ZFC, a notion of forcing $\mathbb{P} = (P, \leq) \in \mathfrak{M}$ is a partial order with a maximum denoted by $1_{\mathbb{P}}$. The elements of P are called conditions and when $p \leq q$ we say that p extends q , or that p is stronger than q . We say that p and q are **compatible** if there is r which extends both of them, otherwise p and q are **incompatible**. If a certain definition of \mathbb{P} does not result in a partial order that has a maximum, we add one artificially. We will also consider only non-trivial notions of forcing, that is to say that every $p \in P$ has two incompatible extensions.

We define by induction the class of \mathbb{P} -names (calculated within \mathfrak{M}):

- (1) $\mathfrak{M}_0^{\mathbb{P}} = \emptyset$;
- (2) $\mathfrak{M}_{\alpha+1}^{\mathbb{P}} = \mathcal{P}(P \times \mathfrak{M}_\alpha^{\mathbb{P}})$;
- (3) $\mathfrak{M}_\delta^{\mathbb{P}} = \bigcup_{\alpha < \delta} \mathfrak{M}_\alpha^{\mathbb{P}}$ for a limit ordinal δ .

Finally the class of \mathbb{P} -names is

$$\mathfrak{M}^{\mathbb{P}} = \bigcup_{\alpha \in \text{Ord}} \mathfrak{M}_\alpha^{\mathbb{P}},$$

where $\mathcal{P}(x)$ denotes the power set of x , and **Ord** denotes the class of ordinals. We will use \dot{x} to denote a \mathbb{P} -name, and \check{x} to denote a canonical name for $x \in \mathfrak{M}$. If G is a \mathbb{P} -generic filter over \mathfrak{M} then \dot{x}^G is the interpretation of \dot{x} by the filter G .

Let $\{\dot{x}_i \mid i \in I\}$ be a class of \mathbb{P} -names in \mathfrak{M} (if it is a proper class we then require it to be definable). We denote by $\{\dot{x}_i \mid i \in I\}^\bullet$ the name $\{(1_{\mathbb{P}}, \dot{x}_i) \mid i \in I\}$. We shall also use $(\dot{x}, \dot{y})^\bullet$ to denote the canonical name for the ordered pair, namely $\{\{\dot{x}\}^\bullet, \{\dot{x}, \dot{y}\}^\bullet\}^\bullet$.

Suppose that π is an automorphism of \mathbb{P} , we may extend π to an automorphism of \mathbb{P} -names by induction,

$$\tilde{\pi}(\dot{x}) = \{(\pi p, \tilde{\pi} \dot{y}) \mid (p, \dot{y}) \in \dot{x}\}.$$

From this point, though, we will only use π to denote the automorphism of \mathbb{P} as well the automorphism of the \mathbb{P} -names. If \mathbb{P} was defined using parameters from A then a permutation of A can be used to define an automorphism of \mathbb{P} . This will be the case in our proofs. It can be shown by induction that if $x \in \mathfrak{M}$ then $\pi \check{x} = \check{x}$ for any $\pi \in \text{Aut}(\mathbb{P})$.

Lemma (The Symmetry Lemma). *Let $\varphi(u_1, \dots, u_n)$ be a formula in the language of set theory, p a condition in \mathbb{P} and $\dot{x}_1, \dots, \dot{x}_n$ are \mathbb{P} -names, and $\pi \in \text{Aut}(\mathbb{P})$. Then*

$$p \Vdash \varphi(\dot{x}_1, \dots, \dot{x}_n) \iff \pi p \Vdash \varphi(\pi \dot{x}_1, \dots, \pi \dot{x}_n)$$

The proof is by induction on the complexity of φ , and can be found in [Kun80, Lemma 7.13(c)].

Suppose that \mathcal{G} is a group of permutations of a set A , and $E \subseteq A$; we define pointwise stabilizer of E as the group $\text{fix}_{\mathcal{G}}(E) = \{\pi \in \mathcal{G} \mid \pi \upharpoonright E = \text{id}_E\}$. If \mathcal{G} acts on \mathbb{P} -names (through its action on \mathbb{P} in most cases) we define the stabilizer of the name \dot{x} as the group $\text{sym}_{\mathcal{G}}(\dot{x}) = \{\pi \in \mathcal{G} \mid \pi \dot{x} = \dot{x}\}$. We omit \mathcal{G} from these notations if it is clear from context.

If \mathcal{G} is a group, we say that $\mathcal{F} \subseteq \mathcal{P}(\mathcal{G})$ is a **filter of subgroups** if whenever $H \in \mathcal{F}$ and $H \leq K$ then $K \in \mathcal{F}$, and \mathcal{F} is closed under finite intersections. We also require the trivial group is not in \mathcal{F} . We say that \mathcal{F} is **normal** if it is closed under conjugation.

If \mathcal{G} is a group of permutations of \mathbb{P} (or acting on it), and \mathcal{F} is a normal filter of subgroups of \mathcal{G} we say that $\dot{x} \in \mathbb{M}^{\mathbb{P}}$ is an **\mathcal{F} -symmetric** name if $\text{sym}(\dot{x}) \in \mathcal{F}$. We define the class of **hereditarily \mathcal{F} -symmetric sets** by induction, \dot{x} is hereditarily \mathcal{F} -symmetric if and only if \dot{x} is \mathcal{F} -symmetric, and for every $(p, \dot{y}) \in \dot{x}$, \dot{y} is hereditarily \mathcal{F} -symmetric. We shall denote $\text{HS}_{\mathcal{F}}$ the class of hereditarily \mathcal{F} -symmetric names, and as usual we will omit \mathcal{F} when it is clear from context.

Let G be a \mathbb{P} -generic filter over \mathbb{M} , and denote by \mathfrak{N} the class $(\text{HS}_{\mathcal{F}})^G = \{\dot{x}^G \mid \dot{x} \in \text{HS}_{\mathcal{F}}\}$. \mathfrak{N} is called a **symmetric extension** (generated by \mathcal{F}) of \mathbb{M} . The following theorem is stated, and its proof can be found in [Jec03].

Theorem. \mathfrak{N} is a transitive model of **ZF** and $\mathbb{M} \subseteq \mathfrak{N} \subseteq \mathbb{M}[G]$. □

For a cardinal κ we denote by DC_κ the **Principle of Dependent Choice for κ** which states that for every non-empty set X , if R is a binary relation such that for every ordinal $\alpha < \kappa$, and every $f: \alpha \rightarrow X$ there is some $y \in X$ such that $f R y$ then there is $f: \kappa \rightarrow X$ such that for every $\alpha < \kappa$, $f \upharpoonright \alpha R f(\alpha)$. We shall abbreviate by $\text{DC}_{<\kappa}$ the assertion $(\forall \lambda < \kappa) \text{DC}_\lambda$.

The axiom of choice implies that DC_κ holds for every κ , and in fact $\forall \kappa. \text{DC}_\kappa$ is equivalent to the axiom of choice. One useful consequence of DC_κ is that for every set X there is either an injection from X into κ or an injection from κ into X . One can find a thorough treatment of DC_κ and related choice principles in [Jec73, Chapter 8].

Lemma 1. *Let \mathcal{F} be a normal filter of subgroups of a group of automorphisms of \mathbb{P} , and $\mathfrak{N} = (\text{HS}_{\mathcal{F}})^G$ be the symmetric extension of \mathbb{M} . If \mathbb{P} is κ -closed and \mathcal{F} is a κ -complete filter then $\mathfrak{N} \models \text{DC}_{<\kappa}$.*

Proof. Let $\lambda < \kappa$, we will show that if $f: \lambda \rightarrow \mathfrak{N}$ is in $\mathbb{M}[G]$ then $f \in \mathfrak{N}$. From this follows that $\mathfrak{N} \models \text{DC}_\lambda$, because if X and R are elements of \mathfrak{N} as in the assumptions of DC_λ , then we can find $f: \lambda \rightarrow X$ in $\mathbb{M}[G]$ (as the latter is a model of AC), and by the proof here we will have that $f \in \mathfrak{N}$.

Let \dot{f}_0 be a name for f and let p be a condition forcing that \dot{f}_0 is a function whose domain is λ and its range is a subset of \mathfrak{N} . Because \mathbb{P} is κ -closed we can extend p to $p_0 \geq p_1 \geq \dots \geq p_\alpha \geq \dots \geq p_\lambda$ such that for all $\alpha < \lambda$, $p_\alpha \Vdash \dot{f}_0(\check{\alpha}) = \dot{t}_\alpha$, where $\dot{t}_\alpha \in \text{HS}$. Then we can define the collection $\{\dot{t}_\alpha \mid \alpha < \lambda\}$ in \mathbb{M} , and take $\dot{f} = \{\dot{t}_\alpha \mid \alpha < \lambda\}^\bullet$. Clearly $\dot{f}^G = f$ whenever $p_\lambda \in G$. We need to show that $\dot{f} \in \text{HS}$, but it is enough to show that $\text{sym}(\dot{f}) \in \mathcal{F}$ because all the names appearing in \dot{f} are taken from HS to begin with. We have that for every $\pi \in \bigcap_{\alpha < \lambda} \text{sym}(\dot{t}_\alpha)$ it holds that $\pi \dot{f} = \dot{f}$, and by κ -completeness of \mathcal{F} we have that the intersection is in \mathcal{F} , and so \dot{f} is in HS as wanted. □

Remember that if A is a set the $|A|$ is the **cardinal number** of A . While in **ZFC** cardinal numbers are all ordinals, without the axiom of choice it is not always the case. We define $|A|$ to be the least ordinal bijectible with A if such ordinal exists, and otherwise $|A|$ is the set of those B which are in bijection with A and have minimal rank with respect to this property. If $|A|$ is a finite ordinal we say that A is

finite, if it is an infinite ordinal we say that $|A|$ is an **aleph number**; in both cases we may say that $|A|$ is a well-ordered cardinal. If $|A|$ is not a well-ordered cardinal we say that A is **not well-orderable**.

For A, B sets we define $|A| \leq |B|$ if and only if there is an injection from A into B , and $|A| \leq^* |B|$ if and only if A is empty or there is a surjection from B onto A . Both relations are reflexive and transitive, but only \leq is provably antisymmetric without the axiom of choice. We also have that $|A| \leq |B|$ implies $|A| \leq^* |B|$. For further analysis of the \leq^* relation see [BM90].

3. EMBEDDING PARTIALLY ORDERED SETS INTO CARDINALS

Let \mathfrak{M} be a countable transitive model of ZFC, κ regular in \mathfrak{M} . Let $(\mathcal{Z}, \leq) \in \mathfrak{M}$ be a partially ordered set. We want to embed (\mathcal{Z}, \leq) into the cardinals of some model, but instead we will embed $(\mathcal{P}(\mathcal{Z}), \subseteq)$. We observe that (\mathcal{Z}, \leq) itself embeds into its power set by the map $z \mapsto \{z' \in \mathcal{Z} \mid z' \leq z\}$, and so it is indeed enough to embed the power set of \mathcal{Z} .

We define $\mathbb{P} = (P, \leq)$ to be the following notion of forcing defined within \mathfrak{M} . $p \in P$ is a partial function $p: (\mathcal{Z} \times \kappa) \times \kappa \rightarrow 2$ such that $|\text{dom } p| < \kappa$. As usual $p \leq q \iff q \subseteq p$. We note that this forcing is κ -closed and therefore does not collapse cardinals smaller than κ^+ . If $\kappa^{<\kappa} = \kappa$, then \mathbb{P} has κ^+ -c.c. and no cardinals are collapsed.

If G is \mathbb{P} -generic over \mathfrak{M} then $\bigcup G = g$ is a total function from $(\mathcal{Z} \times \kappa) \times \kappa$ to 2 in $\mathfrak{M}[G]$. We define the following generic sets and we give them canonical names:

- Let $z \in \mathcal{Z}, \alpha \in \kappa$ we define $r_{z,\alpha} = \{\gamma < \kappa \mid g((z, \alpha), \gamma) = 1\}$, with the canonical name $\dot{r}_{z,\alpha} = \{(p, \check{\gamma}) \mid p((z, \alpha), \gamma) = 1\}$.
- Let $z \in \mathcal{Z}$ we define $R_z = \{r_{z,\alpha} \mid \alpha < \kappa\}$ with the canonical name $\dot{R}_z = \{\dot{r}_{z,\alpha} \mid \alpha < \kappa\}^\bullet$.
- Let $Q \subseteq \mathcal{Z}$ we define $D_Q = \bigcup_{z \in Q} R_z$. We do not give a canonical name to D_Q , because we allow $Q \notin \mathfrak{M}$.

Let \mathcal{G} the group of all permutations of $\mathcal{Z} \times \kappa$ such that for all (z, α) we have $\pi(z, \alpha) = (z, \beta)$ for some β (note that $\pi(z_1, \alpha_1) = (z_1, \beta)$ and $\pi(z_2, \alpha_2) = (z_2, \beta)$ does not imply $\alpha_1 = \alpha_2$). We define the action of \mathcal{G} on \mathbb{P} . If $\pi \in \mathcal{G}$ we define

$$\pi p(\pi(z, \alpha), \gamma) = p((z, \alpha), \gamma).$$

We extend the action of \mathcal{G} to the class of \mathbb{P} -names. We first make the following observation. For all $z \in \mathcal{Z}, \alpha < \kappa$ we have that

$$\pi \dot{r}_{z,\alpha} = \{(\pi p, \pi \check{\gamma}) \mid p((z, \alpha), \gamma) = 1\} = \{(p, \check{\gamma}) \mid p(\pi(z, \alpha), \gamma) = 1\} = \dot{r}_{\pi(z, \alpha)},$$

and for any $z \in \mathcal{Z}$ we have that $\pi \dot{R}_z = \dot{R}_{\pi(z)}$.

Let $I = [\mathcal{Z} \times \kappa]^{<\kappa}$, let \mathcal{F} be the filter generated by $\text{fix}(E)$ for $E \in I$, namely

$$\mathcal{F} = \{H \leq \mathcal{G} \mid \exists E \in I : \text{fix}(E) \leq H\}.$$

It is left to the reader to verify that \mathcal{F} is indeed a normal filter of subgroups and that \mathcal{F} is κ -closed. Let **HS** denote the class of hereditarily symmetric \mathbb{P} -names, and let \mathfrak{N} be HS^G . If $E \in I$ and $\text{fix}(E) \leq \text{sym}(\dot{x})$ we say that E is a **support** of \dot{x} .

We have that $\mathfrak{N} \subseteq \mathfrak{M}[G]$ is a model of ZF. It follows from the κ -closure of \mathbb{P} and \mathcal{F} that the conditions for Lemma 1 hold, and thus $\mathfrak{N} \models \text{DC}_{<\kappa}$. We will see later that the axiom of choice, indeed DC_κ itself does not hold in \mathfrak{N} .

Proposition 2. *For all $z \in \mathcal{Z}$ and $\alpha < \kappa$, $r_{z,\alpha} \in \mathfrak{N}$ and $R_z \in \mathfrak{N}$, and for every $Q \in \mathfrak{M}$ such that $Q \subseteq \mathcal{Z}$ we have that $D_Q \in \mathfrak{N}$.*

Proof. It is enough to observe that because $\pi \dot{r}_{z,\alpha} = \dot{r}_{\pi(z, \alpha)}$ and therefore $\{(z, \alpha)\}$ is a support for $\dot{r}_{z,\alpha}$, and therefore $\dot{r}_{z,\alpha}$ is in **HS**. From this follows that $\dot{R}_z \in \text{HS}$ for

all $z \in \mathcal{Z}$, for $D_Q \in \mathfrak{N}$ one merely has to observe that D_Q is definable from the sets R_z . \square

Two facts which are useful for later are:

Fact 3. *For every $z \in \mathcal{Z}$ we have that R_z can be mapped onto κ , and therefore D_Q can be mapped onto κ for every non-empty Q .*

Proof. The map $r_{z,\alpha} \mapsto \min r_{z,\alpha}$ is well-defined in \mathfrak{N} , and by a simple density argument one can see it is surjective in $\mathfrak{M}[G]$ and therefore in \mathfrak{N} as well. \square

Fact 4. *The function which maps $Q \subseteq \mathcal{Z}$ to D_Q is in \mathfrak{N} (for $Q \in \mathfrak{N}$).*

Proof. Note that $\dot{F} = \left\{ \left(\dot{z}, \dot{R}_z \right)^\bullet \mid z \in \mathcal{Z} \right\}^\bullet$ is in **HS**, since for every z we have that $\text{sym}(\dot{R}_z) = \mathcal{G}$. Let $F = \dot{F}^G$, then for $Q \subseteq \mathcal{Z}$ we have $D_Q = \bigcup_{z \in Q} F(z)$. Therefore whenever $Q \in \mathfrak{N}$, $D_Q \in \mathfrak{N}$ as well. \square

We remark that from a name $\dot{Q} \in \mathbf{HS}$ for a subset of \mathcal{Z} one can give a (relatively) canonical name for D_Q which has the same support as \dot{Q} . However by showing that $D_Q \in \mathfrak{N}$ is definable from $Q \in \mathfrak{N}$ we in fact prove that there is such name.

Proposition 5. *For every $z \in \mathcal{Z}$ we have that R_z cannot be well-ordered. Therefore the axiom of choice fails in \mathfrak{N} .*

Proof. We know that R_z can be mapped onto κ , therefore it would suffice to show that there is no injection from κ into R_z . If R_z could have been well-ordered such surjection could have been reversed to an injection from κ .

Assume towards contradiction that $p \Vdash \ulcorner \dot{f}: \dot{\kappa} \rightarrow \dot{R}_z \text{ is injective} \urcorner$, for some $\dot{f} \in \mathbf{HS}$. Let E be a support for \dot{f} , and let $q \leq p$ be such that for some $\alpha < \kappa$ such that $(z, \alpha) \notin E$ we have that for some $\tau < \kappa$, $q \Vdash \dot{f}(\check{\tau}) = \dot{r}_{z,\alpha}$.

We can now find $\beta \neq \alpha$ such that $(z, \beta) \notin E$ and for all γ , $((z, \beta), \gamma) \notin \text{dom } q$. We define the following $\pi \in \mathcal{G}$: $\pi(z, \alpha) = (z, \beta)$, $\pi(z, \beta) = (z, \alpha)$ and $\pi(x, y) = (x, y)$ otherwise. Clearly $\pi \in \text{fix}(E)$ and therefore $\pi \dot{f} = \dot{f}$. By the symmetry lemma we have that

$$\pi q \Vdash \dot{f}(\check{\tau}) = \dot{r}_{z,\beta}.$$

If q and πq are compatible then q has an extension which forces that \dot{f} is not a function, which is a contradiction. And indeed suppose $((t, \varepsilon), \delta) \in \text{dom } q \cap \text{dom } \pi q$. If $t \neq z$ then $\pi(t, \varepsilon) = (t, \varepsilon)$ and by the definition of πq we have

$$\pi q((t, \varepsilon), \delta) = \pi q(\pi(t, \varepsilon), \delta) = q(t, \varepsilon, \delta),$$

as wanted. Otherwise $t = z$ and then $\varepsilon \neq \beta$, otherwise the tuple would not be in $\text{dom } q$, and by the same argument $\varepsilon \neq \alpha$ as in such case the tuple would not be in $\text{dom } \pi q$. Therefore the conditions are compatible as wanted.

Therefore there is no $\dot{f} \in \mathbf{HS}$ such that \dot{f} is a name for an injection from κ into R_z , as wanted. \square

We have in fact shown that κ and R_z have incomparable cardinalities in \mathfrak{N} , and therefore DC_κ fails as promised.

Theorem 6. *Let $Q, T \subseteq \mathcal{Z}$ in \mathfrak{N} . If $Q \not\subseteq T$ then $\mathfrak{N} \models |D_Q| \not\leq |D_T|$ and $|D_Q| \not\leq^* |D_T|$.*

Proof. If There had been an injection from D_Q into D_T then there would have been a surjection from D_T onto D_Q . It is therefore sufficient to argue for the $\not\leq^*$ case.

Let \dot{Q} and \dot{T} be names for Q and T respectively, both in **HS**, and let \dot{D}_Q and \dot{D}_T be names in **HS** for D_Q and D_T respectively.

Suppose that $p \Vdash \ulcorner \dot{f}: \dot{D}_T \rightarrow \dot{D}_Q \text{ is surjective} \urcorner$ for some $\dot{f} \in \mathbf{HS}$. Assume that $p \Vdash \dot{Q} \not\subseteq \dot{T}$ as well. Let E be a support for the names $\dot{f}, \dot{Q}, \dot{T}, \dot{D}_Q, \dot{D}_S$.

Let $q \leq p$ be an extension such that there is some $z \in \mathcal{Z}$ and $\alpha, \beta < \kappa$ such that $(z, \alpha) \notin E$ and $q \Vdash \dot{z} \in \dot{Q} \setminus \dot{T} \wedge \dot{f}(\dot{r}_{t,\delta}) = \dot{r}_{z,\alpha}$. Note that $z \neq t$ as $q \Vdash \dot{r}_{t,\delta} \in \dot{D}_T$ and therefore $q \Vdash \dot{t} \in \dot{T}$. Let $\beta \neq \alpha$ be such that $(z, \beta) \notin E$ and there is no $\gamma < \kappa$ for which $((z, \beta), \gamma) \in \text{dom } q$. We define π to be the permutation such that $\pi(z, \alpha) = (z, \beta), \pi(z, \beta) = (z, \alpha)$ and $\pi(x, y) = (x, y)$ otherwise. We have that $\pi \in \text{fix}(E)$ and therefore all the names of interest are not changed by π .

We have that $\pi q \Vdash \dot{f}(\dot{r}_{t,\delta}) = \dot{r}_{z,\beta}$. However a simple verification as in the proof of [Proposition 5](#) shows that q and πq are compatible and therefore q has an extension which forces \dot{f} is not a function, which is a contradiction.

We finish the proof with the remark that if $p \nVdash \dot{Q} \subseteq \dot{T}$ then either it forces $\dot{Q} \subseteq \dot{T}$, in which case the assumption $\dot{Q} \not\subseteq \dot{T}$ fails; or that p does not decide the statement $\dot{Q} \subseteq \dot{T}$, but it is impossible for p to have an extension which forces $\dot{Q} \not\subseteq \dot{T}$ by the above argument. Therefore p has already decided that $\dot{Q} \subseteq \dot{T}$. \square

4. EMBEDDING A PROPER CLASS

In this section we extend the result by Roguski ([\[Rog90\]](#)) in which he proves the following theorem:

Theorem (Roguski). *Let \mathfrak{M} be a countable transitive model of ZFC, and (I, \preceq) a partially ordered class such that I, \preceq are both classes of \mathfrak{M} and every initial segment of (I, \preceq) belongs to \mathfrak{M} . Then there is a countable transitive model \mathfrak{N} for ZF, which is a symmetric extension of \mathfrak{M} and a class $\{S_i \mid i \in I\}$ in \mathfrak{N} such that for all $i, j \in I$,*

$$i \preceq j \leftrightarrow \mathfrak{N} \models |S_i| \leq |S_j|.$$

From this theorem he draws the consistency of a proper class of pairwise incomparable cardinals. Using [Theorem 7](#) we will show that the result can be carried out in such way that $\mathbf{DC}_{<\kappa}$ is preserved for a fixed κ , and that we may replace $|S_i| \leq |S_j|$ by $|S_i| \leq^* |S_j|$.

Let \mathfrak{M} be a countable transitive model of $\mathbf{ZFC} + \mathbf{GCH}$, κ a regular cardinal in \mathfrak{M} and (I, \preceq) a partially ordered class in \mathfrak{M} such that every initial segment of I is a set of \mathfrak{M} . Without loss of generality we may assume that there is a class in \mathfrak{M} which well-orders I , for otherwise we can force such class without adding sets. Therefore we may assume that $I \subseteq \mathbf{Ord}^{\mathfrak{M}}$. In this section we shall prove the following theorem:

Theorem 7. *There is a symmetric extension \mathfrak{N} of \mathfrak{M} such that $\mathfrak{N} \models \mathbf{ZF} + \mathbf{DC}_{<\kappa}$ in which (I, \preceq) can be embedded into the cardinals with the order \leq or with the order \leq^* .*

By embedding $(\mathcal{P}(I) \cap \mathfrak{M}, \subseteq)$ into the cardinals of the symmetric extension we will assure that I has been embedded into it using the same argument as in the previous section. Note that if I is actually a set in \mathfrak{M} then [Theorem 6](#) proves the claim, so we may assume that I is a proper class of \mathfrak{M} . We aim to mimic the previous proof therefore for every $i \in I$ we shall add generic subsets to a regular cardinal. In order to preserve $\mathbf{DC}_{<\kappa}$ we require the forcing to be κ -closed, so we will only add subsets to cardinals above κ . We will assume that I is a class of regular cardinals and $\min I \geq \kappa$.

We define the forcing in \mathfrak{M} . For every $i \in I$ let $\mathbb{P}_i = (P_i, \leq)$ be the forcing which adds i Cohen subsets to i , namely $p \in P_i$ is a function from $i \times i$ to 2 such that $|\text{dom } p| < i$, and $p \leq q$ if and only if $q \subseteq p$. Let \mathbb{P} be the Easton support product $\prod_{i \in I} \mathbb{P}_i$. We shall denote $\mathbb{P}^{\leq i}$ the Easton support product of \mathbb{P}_j for $j \leq i$. This is a

product of κ -closed forcings and therefore it is κ -closed, and we also point out that by assuming GCH it does not change cofinalities.

The conditions in \mathbb{P} are functions such that $p(i)$ is a condition in P_i , and for all $i \in I$ we have $|\{j \leq i \mid p(j) \neq 1_{\mathbb{P}_j}\}| < i$. Alternatively we may think about the conditions as functions from $I \times \text{Ord} \times \text{Ord}$ to $\{0, 1\}$ such that if (i, α, β) is in $\text{dom } p$ then $\beta, \alpha < i$, and for every $i \in I$, $|\{(j, \alpha, \beta) \mid (j, \alpha, \beta) \in \text{dom } p \wedge j \leq i\}| < i$. We will identify $\mathbb{P}^{\leq i}$ with those $p \in \mathbb{P}$ such that $\text{dom } p \subseteq (i+1) \times i \times i$.

If G is a \mathbb{P} -generic class over \mathfrak{M} then in $\mathfrak{M}[G]$ it defines i new subsets for every (regular) cardinal in I , and $\mathfrak{M}[G]$ is a model of ZFC. Note that as before $\bigcup G = g$ is a class function $g: I \times \text{Ord} \times \text{Ord} \rightarrow 2$. We define the following sets from G and give them canonical names:

- Let $i \in I$ and $\alpha < i$ then $r_{i,\alpha} = \{\gamma < i \mid g(i, \alpha, \gamma) = 1\}$ is given the name

$$\dot{r}_{i,\alpha} = \{(p, \gamma) \mid p(i, \alpha, \gamma) = 1 \wedge p \in \mathbb{P}^{\leq i}\}.$$

- Let $i \in I$ we define $R_i = \{r_{i,\alpha} \mid \alpha < i\}$ with the name

$$\dot{R}_i = \{\dot{r}_{i,\alpha} \mid \alpha < i\}^\bullet.$$

- For a set $Q \subseteq I$ denote $D_Q = \bigcup_{i \in Q} R_i$. Of course Q might be generic, and as before we do not give a name for D_Q .

We shall now proceed to define the symmetric extension. First we define \mathcal{G} to be a group of automorphisms of \mathbb{P} , while this group will be a proper class each permutation will only move a set. We say that $\pi \in \mathcal{G}$ if π is a permutation of $I \times \text{Ord} \times \text{Ord}$ such that the following holds:

- (1) For all $i \in I$, if $(i, \alpha, \gamma) \in \text{dom } \pi$ then $\alpha, \gamma < i$;
- (2) whenever $\pi(i, \alpha, \gamma) = (i', \alpha', \gamma')$ we have $i = i'$, $\alpha' < i$, $\gamma = \gamma'$;
- (3) $\mathbb{D}_\pi = \{(i, \alpha, \gamma) \mid \pi(i, \alpha, \gamma) \neq (i, \alpha, \gamma)\}$ is a set in \mathfrak{M} ;
- (4) and for every $i \in I$, $|\{(i, \alpha, \gamma) \mid (i, \alpha, \gamma) \in \mathbb{D}_\pi\}| < i$.

We define the action of \mathcal{G} on \mathbb{P} as before,

$$\pi p(\pi(i, \alpha, \gamma)) = p(i, \alpha, \gamma).$$

For each $i \in I$ we define $\mathcal{G}_i = \{\pi \in \mathcal{G} \mid \mathbb{D}_\pi \subseteq (i+1) \times i \times i\}$. Then \mathcal{G}_i is a group, $i \leq j$ implies $\mathcal{G}_i \leq \mathcal{G}_j$ and $\bigcup_{i \in I} \mathcal{G}_i \simeq \mathcal{G}$. We define $G_{\leq i} = G \cap \mathbb{P}^{\leq i}$. We observe that $\mathfrak{M}^{\mathbb{P}^{\leq i}} \subseteq \mathfrak{M}^{\mathbb{P}^{\leq j}}$ whenever $i \leq j$, and that if $\dot{x} \in \mathfrak{M}^{\mathbb{P}^{\leq i}}$ then $\dot{x}^{G_i} = \dot{x}^{G_j}$ as well.

Let $K_i = [(I \cap i^+) \times i \times i]^{<\kappa}$, and let \mathcal{F}_i be the κ -complete filter of subgroups of \mathcal{G}_i generated by $\text{fix}(E)$ for $E \in K_i$,

$$\mathcal{F}_i = \{H \leq \mathcal{G}_i \mid \exists E \in K_i : \text{fix}(E) \leq H\}.$$

We have that for $i \leq j$, $\mathcal{F}_i \subseteq \mathcal{F}_j$. For every $i \in I$ let HS_i be $\text{HS}_{\mathcal{F}_i} \subseteq \mathfrak{M}^{\mathbb{P}^{\leq i}}$. Let $\mathcal{F} = \bigcup_{i \in I} \mathcal{F}_i$, then $\text{HS} = \text{HS}_{\mathcal{F}} = \bigcup_{i \in I} \text{HS}_i$.

It is a standard way to define $\mathfrak{M}[G]$ as the union of $\bigcup_{i \in I} \mathfrak{M}[G_i]$. For every $i \in I$ we define $\mathfrak{N}_i = (\text{HS}_i)^{G_i} \subseteq \mathfrak{M}[G_i]$ to be a symmetric extension of \mathfrak{M} . Then for $i \leq j$ we have $\mathfrak{N}_i \subseteq \mathfrak{N}_j$, and every \mathfrak{N}_i has the same ordinals (and initial ordinals) and satisfies $\text{ZF} + \text{DC}_{<\kappa}$.

Let $\mathfrak{N} = \bigcup_{i \in I} \mathfrak{N}_i$. We first observe the following equality holds,

$$\mathfrak{N} = \bigcup_{i \in I} \mathfrak{N}_i = \bigcup_{i \in I} (\text{HS}_i)^{G_i} = \bigcup_{i \in I} (\text{HS}_i)^G = \left(\bigcup_{i \in I} \text{HS}_i \right)^G = \text{HS}^G.$$

Then \mathfrak{N} is a model of $\text{ZF} + \text{DC}_{<\kappa}$. The model satisfies $\text{DC}_{<\kappa}$ because every \mathfrak{N}_i does. We shall see that $r_{i,\alpha}, R_i$ and D_Q (for $Q \in \mathfrak{N}$) are in \mathfrak{N} , and that the class $\{D_Q \mid Q \in \mathcal{P}(I) \cap \mathfrak{N}\}$ is a class of \mathfrak{N} . Note that the equality above tells us that $x \in \mathfrak{N}$ if and only if there exists $\dot{x} \in \text{HS}$ such that $\dot{x}^G = x$, and therefore there

exists $E \in [I \times \mathbf{Ord} \times \mathbf{Ord}]^{<\kappa} \cap \mathfrak{M}$ such that $\text{fix}(E) \leq \text{sym}(\dot{x})$. As before we say that E is a support of \dot{x} .

Proposition 8. *For all $i \in I, \alpha < i$ we have $r_{i,\alpha} \in \mathfrak{N}$, $R_i \in \mathfrak{N}$. Furthermore $i \mapsto R_i$ is definable, and therefore whenever $Q \subseteq I$ is a set in \mathfrak{N} then D_Q is in \mathfrak{N} .*

Proof. It is immediate that $\{(i, \alpha, 0)\}$ is a support of $\dot{r}_{i,\alpha}$ and that $\dot{r}_{i,\alpha} \in \mathbf{HS}_j$ for any $j > i$. It therefore follows that \emptyset is a support of \dot{R}_i which is also in \mathbf{HS}_j . From that it follows that $r_{i,\alpha}$ and R_i are both in \mathfrak{N} for any i and α .

Consider the class name $\dot{F} = \left\{ (\dot{i}, \dot{R}_i)^\bullet \mid i \in I \right\}^\bullet$. For all $j \in I$ we have that the name $\dot{F}_j = \left\{ (\dot{i}, \dot{R}_i)^\bullet \mid i \leq j \right\}^\bullet$ is a symmetric name in \mathbf{HS}_j , and $\text{sym}_{\mathcal{G}_j}(\dot{F}_j) = \mathcal{G}_j$. Therefore $\dot{F} = \bigcup_{i \in I} \dot{F}_i$ is a symmetric class in \mathbf{HS} , and its interpretation $F = \dot{F}^G$ is a class of \mathfrak{N} . From this follows that whenever $Q \subseteq I$ is a set in \mathfrak{N} then

$$D_Q = \bigcup_{i \in Q} F(i) = \{x \mid \exists i \in Q : x \in F(i)\},$$

is in \mathfrak{N} as promised. \square

We observe that as in [Fact 3](#) every R_i can be mapped onto i . While the map $r_{i,\alpha} \mapsto \min r_{i,\alpha}$. Obviously there is no R_i that can be mapped onto i^+ . Furthermore the proof that $\kappa \not\leq |R_i|$ is the same as the proof of [Proposition 5](#).

Proposition 9. *For every Q, T subsets of I , $\mathfrak{N} \models Q \subseteq T \leftrightarrow |D_Q| \leq^* |D_T|$.*

Proof. If $Q \subseteq T$ then $D_T \subseteq D_Q$ and the result is trivial. Suppose that \dot{Q} and \dot{T} are names in \mathbf{HS} for the sets Q, T respectively. Assume towards contradiction that $p \Vdash \dot{Q} \not\subseteq \dot{T} \wedge \ulcorner \dot{f} : \dot{D}_T \rightarrow \dot{D}_Q \text{ is a surjection} \urcorner$ where $\dot{f}, \dot{D}_Q, \dot{D}_T \in \mathbf{HS}$ and \dot{D}_Q, \dot{D}_T are names for D_Q and D_T respectively.

The proof of [Theorem 6](#) applies here completely by noting that there is some $i \in I$ such that the entire proof is actually carried in $\mathbb{P}^{\leq i}$. \square

This concludes the proof of [Theorem 7](#), as taking $S_i = \{j \leq i\}$ guarantees that $|S_i| \leq^* |S_j| \leftrightarrow |S_i| \leq |S_j| \leftrightarrow i \leq j$.

5. EXTENSIONS OF THE THEOREM

We draw two corollaries from the theorem and show the independence of two choice principles from \mathbf{DC}_κ (for any κ). When a choice principle is not provable by \mathbf{DC}_κ , for any κ , it hints us that it may be equivalent to the axiom of choice, or that it is “orthogonal” to \mathbf{DC}_κ -like principles.

Theorem 10. *For every cardinal μ it is consistent with $\mathbf{ZF} + \mathbf{DC}_\mu$ that there is an antichain of cardinals which is a proper class, and for every ordinal α there is a decreasing sequence of cardinals of order type α^* .*

Proof. Let $\kappa > \mu$ and consider the model from Section 4 in which we embed the ordinals with the discrete order into the cardinals of \mathfrak{N} . We have that $\mathfrak{N} \models \mathbf{DC}_{<\kappa}$ and therefore $\mathfrak{N} \models \mathbf{DC}_\mu$. For any $\alpha \in \mathbf{Ord}$ we have R_α as defined as in the proof of [Theorem 7](#), then $\{|R_\alpha| \mid \alpha \in \mathbf{Ord}\}$ is an antichain in both \leq^* and in \leq .

Let α be an ordinal, and define for $\beta < \alpha$ the set $D_{\beta^*} = \bigcup \{R_\gamma \mid \beta \leq \gamma < \alpha\}$. By [Proposition 9](#) we have that $\{|D_{\beta^*}| \mid \beta < \alpha\}$ is a decreasing chain of cardinals in both \leq and \leq^* . \square

Note that it is impossible to find a decreasing sequence of order type \mathbf{Ord}^* . Any set whose cardinality is the maximum of such sequence must have a proper class of different subsets, which is a contradiction to the power set axiom.

Finally we will show that the constructive set-theory oriented axiom known as **WISC** is independent from **ZF**. This result was known due to van den Berg (see [van12])². van den Berg's proof assumes some very large cardinals, and we improve it by removing this additional assumption and by showing the compatibility of this failure with DC_μ for arbitrary μ .

The principle **WISC** (Weakly Initial Set Cover) can be formulated as follows: *For every set X there is a set Y , such that whenever Z is a set and $f: Z \rightarrow X$ is a surjection then there is $q: Y \rightarrow Z$ such that $f \circ q$ is onto X .* This formulation is due to François Dorais (see [Rob13] for more details).

We will now show that the model from Theorem 10 satisfies $\neg\text{WISC}$. Recall that for all α , R_α can be mapped onto κ . We will show that for κ there is no Y as in the requirement of **WISC**.

Theorem 11. *Let \mathfrak{N} be the model from Theorem 10, then for every set $Y \in \mathfrak{N}$, there is $\alpha \in \mathbf{Ord}$ such that R_α can be mapped onto κ by some function h , but every function $f: Y \rightarrow R_\alpha$ has range of cardinality $< \kappa$. Therefore there is no such f for which $h \circ f$ is onto κ . In particular $\mathfrak{N} \models \neg\text{WISC}$.*

Proof. We will show that in \mathfrak{N} for every Y there is some α such that any $f: Y \rightarrow R_\alpha$ must satisfy $|\text{rng } f| < \kappa$, and therefore it is impossible that any composition of f with a function from R_α is onto κ .

Let $Y \in \mathfrak{N}$ be any set, and let $\alpha \in \mathbf{Ord}$ be such that for some $\beta < \alpha$ we have $\dot{Y} \in \text{HS}_\beta$. This means that any condition which appears in \dot{Y} appears in $\mathbb{P}^{\leq \beta}$. Suppose that $p \Vdash \dot{f}: \dot{Y} \rightarrow \dot{R}_\alpha$, and $\dot{f} \in \text{HS}$. If $p \Vdash |\text{rng } \dot{f}| < \kappa$ then we are done, assume that this is not the case, and that $p \Vdash |\text{rng } \dot{f}| \not< \kappa$.

Let $E \in [\mathbf{Ord} \times \mathbf{Ord} \times \mathbf{Ord}]^{<\kappa}$ be a support for \dot{f}, \dot{Y} (recall that \dot{R}_α is supported by any set). Let $q \leq p$ be such that there is $\delta < \alpha$ such that for all $\gamma < \alpha$, $(\alpha, \delta, \gamma) \notin E$, and for some \dot{y} we have that $q \Vdash \dot{f}(\dot{y}) = \dot{r}_{\alpha, \delta}$. We can now find $\tau \neq \delta$ such that $(\alpha, \tau, \gamma) \notin E \cup \text{dom } q$ for any $\gamma < \alpha$. Let π be the permutation in \mathcal{G} defined as follows: $\pi(\alpha, \delta, \gamma) = (\alpha, \tau, \gamma)$; $\pi(\alpha, \tau, \gamma) = (\alpha, \delta, \gamma)$; and $\pi(x, y, z) = (x, y, z)$ otherwise.

As \dot{y} is a name appearing in \dot{Y} , and thus $\dot{y} \in \text{HS}_\beta$, we have that any condition in \dot{y} appears in $\mathbb{P}^{\leq \beta}$. This means that for any permutation in \mathcal{G} which does not move any condition in $\mathbb{P}^{\leq \beta}$ will not move \dot{y} either, in particular this is true for π defined above.

We have that $\pi q \Vdash \dot{f}(\dot{y}) = \dot{r}_{\alpha, \tau}$, and as in the proof of Theorem 6 we have that q and πq are compatible which is a contradiction, and the conclusion follows as wanted. \square

Therefore for every μ , **WISC** is unprovable from $\text{ZF} + \text{DC}_\mu$. This extends the results by Rathjen which establish the independence of a slightly stronger choice principle from **ZF** by a similar method as van den Berg (see [Rat06]).

6. ACKNOWLEDGEMENTS

The author wishes to thank Uri Abraham and Matatyahu Rubin for many conversations which helped to shape this paper, and for their help in revising the manuscript. He also thanks Andrés Caicedo for introducing him to the problem of antichains of cardinals, and for additional suggestions. And to David M. Roberts for his help with the parts regarding **WISC**.

²van den Berg names this axiom *the Axiom of Multiple Choice*, an unfortunate name as it is already the name of a relatively known choice principle.

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